Chapter 10 Material

The solutions to problems from Chapter 10 are presented. Note, in particular, that the solution to Problem 10.14 develops the theory of first-degree price discrimination in a B2C context.

10.1 In Table S10.1, I give the utility level of each bundle, for each of the three consumers. From this table, it is clear that consumer 1 will choose bundle #2, consumer 2 will choose bundle #1, and consumer 3 will choose bundle #3.

<table>
<thead>
<tr>
<th>bundle 1</th>
<th>consumer 1</th>
<th>consumer 2</th>
<th>consumer 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.347</td>
<td>16.000</td>
<td>2.227</td>
<td></td>
</tr>
<tr>
<td>0.458</td>
<td>12.500</td>
<td>3.114</td>
<td></td>
</tr>
<tr>
<td>0.112</td>
<td>0.625</td>
<td>3.833</td>
<td></td>
</tr>
</tbody>
</table>

*Table S10.1. Problem 10.1: Utility levels for three consumers and three bundles.*

10.2 Equating bangs for the buck gives

\[
\frac{6}{b} = \frac{1}{1.2} = \frac{3}{c} = \frac{1}{3} = \frac{1}{s} = \frac{1}{4} \quad \text{or} \quad \frac{1.2b}{6} = \frac{3c}{3} = 4s \quad \text{or} \quad \frac{b}{5} = c = 4s.
\]

Since utility is strictly increasing in all three goods, we know that the consumer will spend all her wealth, and so we also have

\[1.2b + 3c + 4s = 20.\]

Doing a bit of algebra, this gives us

\[6c + 3c + c = 20 \quad \text{or} \quad 10c = 20 \quad \text{or} \quad c = 2,
\]

hence,

\[b = 5c = 10 \quad \text{and} \quad s = \frac{c}{4} = 0.5.
\]
That is, the consumer will purchase 10 loaves of bread (cost $12.00), two kilos of cheese (cost $6.00), and half a kilo of salami (cost $2.00).

10.3 This is a bang-for-the-buck problem with money-left-over in the utility function, taking the form \( \ldots + m \). The problem is answered (tentatively) by assuming that the consumer will have some money left over, in which case you equate the bang-for-the-buck of each non-money commodity to the bang-for-the-buck of money, which is 1.

First record the bangs for the buck of bread, cheese, and salami: They are

\[
\text{BfB}_b = \frac{6}{b^{1.20}}, \quad \text{BfB}_c = \frac{1}{c^{1.2}}, \quad \text{and} \quad \text{BfB}_s = \frac{1}{s^{1.4}}.
\]

Next take each non-money commodity in turn, and equate its bang-for-the-buck to 1. This is

\[
\frac{6}{b^{1.20}} = 1 \quad \text{or} \quad b = \frac{6}{1.2} = 5,
\]

\[
\frac{1}{c^{1.2}} = 1 \quad \text{or} \quad c = 2/3,
\]

and

\[
\frac{1}{s^{1.4}} = 1 \quad \text{or} \quad s = 1/4.
\]

Now to check the hypothesis that this leaves the consumer with some money:

\[
(1.2)(5) + (3)(2/3) + (4)(1/4) = 6 + 2 + 1 = 9.
\]

Since the consumer starts with $160, spending $9 leaves her with $151. Done.

10.4 (a) Begin by hypothesizing that, at the solution, all three goods will be consumed in strictly positive amounts. Then, their bangs for the buck must be equal, or

\[
\frac{8}{b+2} \times \frac{1}{1} = \frac{6}{c+1} \times \frac{1}{2} = \frac{2 \times 2}{2s+1} \times \frac{1}{4}.
\]
Flipping these fractions over and simplifying gives

\[
\frac{b + 2}{8} = \frac{c + 1}{3} = 2s + 1.
\]

Since utility is strictly increasing in all three goods, we know that the budget equation holds at the solution, or

\[
b + 2c + 4s = 18.
\]

Using the equal-bangs-for-the-buck equations, we find that

\[
2c = \frac{3b}{4} + 1.5 - 2 = \frac{3b}{4} - 0.5 \quad \text{and} \quad 4s = \frac{b}{4} + 0.5 - 2 = \frac{b}{4} - 1.5,
\]

therefore the budget equation can be rewritten as

\[
b + 0.75b - 0.5 + 0.25b - 1.5 = 2b - 2 = 18,
\]

which gives \( b = 10 \). From this, we calculate

\[
c = \frac{3}{8} \times 12 - 1 = \frac{28}{8} = 3.5 \quad \text{and} \quad s = \frac{1}{2} \times \left( \frac{12}{8} - 1 \right) = 0.25,
\]

which is the answer.

(b) If the consumer has the same utility function but only $6.50 to spend, and we begin with the hypothesis that she consumes all three goods in strictly positive amounts, we have the same equal-bangs-for-the-buck equations, but the budget equation becomes

\[
b + 2c + 4s = 2b - 2 = 6.50,
\]

or \( b = 4.25 \). This gives

\[
c = \frac{3 \times 4.25}{8} - \frac{1}{4} = \frac{10.75}{8} \quad \text{and} \quad s = \frac{3.75}{16} - \frac{1.5}{4} = -\frac{2.25}{16}.
\]

This gives us a negative value for \( s \), which violates the nonnegativity constraint. So the hypothesis that all three variables are strictly positive is false. We hypothesize next that \( s \) equals 0 at the solution but \( b \) and \( c \) are strictly positive. So the bangs for the buck of \( b \) and \( c \) must be equal, and \( b + 2c = 6.50 \).
is the budget equation. Substituting \(0.75b - 0.5\) for \(2c\) (from the equal bangs-for-the-buck condition), this is

\[
b + 0.75b - 0.5 = 6.50 \quad \text{or} \quad b = \frac{7}{1.75} = 4.
\]

Corresponding to this is

\[
c = 3 \times \left(\frac{4 + 2}{8}\right) - 1 = \frac{10}{8}.
\]

To check that this is the answer, I evaluate the bangs for the buck of bread and cheese at the levels \(4\) and \(\frac{10}{8}\), respectively, and compare with the bang for the buck of salami at \(s = 0\). According to my calculator, these are \(\frac{8}{6} = 1.333\), \(3/(\frac{10}{8} + 1) = 3/(\frac{18}{8}) = \frac{24}{18} = \frac{4}{3} = 1.333\), and \(1\), respectively, so these are okay: The bangs for the buck of the commodities being consumed are equal, and they exceed the bang for the buck of the good not being consumed. Finally, just to check my math, I check the budget equation: \(4\) loaves of bread costs \$4, and \(\frac{10}{8}\) of a kilo of cheese costs \(\frac{20}{8} = \$2.50\), so the total expenditure is indeed \$6.50. We have the answer.

(c) Now the consumer has money left over in her utility function, entering as \(\ldots + m\). So begin by assuming the consumer ends with money left over. The bang for the buck for money left over is \(1\). This is equal to the bang for the buck of bread if \(8/(b + 2) = 1\), or \(b = 6\). It is equal to the bang for the buck of cheese if \(3/(c + 1) = 1\), or \(c = 2\). And it is equal to that of salami if \(1/(2s + 1) = 1\), which is true where \(s = 0\). This bundle costs \(6 \times \$1 + 2 \times \$2 + 0 \times \$4 = \$10\), so as long as the consumer has at least \$10 in her pocket, this is what she consumes: six loaves of bread, 2 kilos of cheese, and no salami, leaving the store with \$10 less than she entered with. This covers the case of initial wealths of \$50, \$500, and \$18. But, if the consumer begins with \$6.50 only, then she leaves the store having spent everything for lunch. And the answer is precisely the answer obtained for the second part of \(b\): four loaves of bread and \(\frac{10}{8}\) kilos of cheese.

10.5 (a) The point of this problem is to look at a utility function where you can’t be sure that the consumer will consume strictly positive amounts of all the commodities. Because the bread part of the utility function is \(10 \ln(b)\), we can be sure that \(b\) will be strictly positive. But the cheese and salami parts are \(\ln(c + 1)\) and \(.5 \ln(s + 4)\), and it isn’t a calamity at all to consume zero cheese or salami.
To solve this problem, we begin with the temporary hypothesis that everything will be consumed in strictly positive amounts. Then equal bangs-for-the-buck give

\[
\frac{10}{2b} = \frac{1}{5(c + 1)} = \frac{.5}{10(s + 4)},
\]

or

\[
.2b = 5c + 5 = 20s + 80.
\]

Since utility is strictly increasing in all three commodities, there is no question that the consumer will spend all her money, or

\[
2b + 5c + 10s = 60.
\]

We substitute \(.2b - 5\) for \(5c\) and \(.1b - 40\) for \(10s\) in the budget equation (using the equal bfb s) to see that \(.2b - 5 = 5c\) and \(.1b - 40 = 10s\), and we get

\[
2b + .2b - 5 + .1b - 40 = 83 \quad \text{or} \quad 2.3b = 128 \quad \text{or} \quad b = 55.65.
\]

Since \(.2b = 11.13 = 5c + 5\), this is \(c = 6.13/5\), and since \(.2b = 11.13 = 20s + 80\), this is \(s = -68.87/20\). Which certainly isn’t the answer; salami consumption can’t be negative.

So we guess next that the answer will entail \(s = 0\) and \(b\) and \(c\) both strictly positive. Equal bfb s for bread and salami tell us that \(.2b = 5c + 5\), and thus \(2b + 5c = 83\) becomes \(2b + .2b - 5 = 83\) or \(2.2b = 88\) or \(b = 40\), and thus \(.2b = 8 = 5c + 5\), which gives \(c = 3/5 = .6\). Just to check, the bfb for bread at 40 loaves is \(10/80 = .125\), the bfb for cheese at .6 of a kilo is \(1/(5(.6 + 1)) = 1/8 = .125\), the bfb for salami at 0 kilos is \(.5/(10 \times 4) = .5/40 = 1/80 = .0125\), and total expenditures are \$2 \times 40 + $5 \times .6 = $83\). The bfb s of the two commodities consumed in strictly positive amount are equal, they exceed the bfb of the commodity that is not consumed, and the consumer spends all her money. That’s the answer!

(b) In part b, the consumer has a money-left-over utility function. The bangs-for-the-buck of bread and cheese are infinite at \(b = 0\) and \(c = 0\), so we know the consumer will buy positive amounts of those two commodities. But the bang-for-the-buck of salami at \(s = 0\) is \((1/(s + 4))/10 = 1/40\), which is much less than the bang-for-the-buck of money-left-over (which is 1), so the
consumer will not buy any salami. Assuming she ends up with money-left-over, her purchases of bread and cheese are where their bfbs are 1, or

$$\frac{10}{b} \frac{1}{2} = \frac{5}{b} = 1 \quad \text{and} \quad \frac{11}{c} \frac{1}{5} = \frac{1}{5c} = 1,$$

which gives us $b = 5$ and $c = 0.2$. $b = 2$ costs $10$, while $c = 0.2$ costs $1$, so as long as the consumer begins with at least $11$, this is the answer; it is certainly the answer if the consumer has $83$ to spend (so she leaves with $72$ in her pocket). But if she only has $6.60$ to spend, she will spend it all on bread and cheese: The equal bfbs and budget constraint equations are

$$\frac{5}{b} = \frac{1}{5c} \quad \text{and} \quad 2b + 5c = 6.60,$$

which (you do the math) have solution $b = 3$ and $c = 0.12$.

10.6 (a) The utility that the consumer obtains from the bundle ($5.00, 1$ stick) is $u(1, 5) = 4 - 1 + 5 = 8$. Hence, $m^*$ must solve

$$u(1.5, m^*) = 4 \times 1.5 - 1.5^2 + m^* = 8 \quad \text{or} \quad 6 - 2.25 + m^* = 8 \quad \text{or} \quad m^* = 8 - 6 + 2.25 = 4.25.$$

(b) The utility of ($5.00, 1$ stick) = 8, hence, the indifference curve through this point is the set of points $(m, c)$ such that $4c - c^2 + m = 8$, or $m = c^2 - 4c + 8$. And the utility of ($6.00, 1$ stick) = 9, so the indifference curve through this point is the set of points $(m, c)$ such that $m = c^2 - 4c + 9$. To graph the two indifference curves, we have to graph these two parabolas. Figure S10.1 depicts the two indifference curves. Notice that the direction of increasing preference is north only (more money left over is always better than less); past 2 sticks of cotton candy, more cotton candy (holding money fixed) takes the consumer on to lower and lower indifference curves.

10.7 Dot #1 is the worst, #2 is third best, #3 is the best, #4 is second best, and #5 is fourth best.

10.8 The budget set is depicted in Figure S10.2. Note that we have a standard triangle-shaped budget set, where we find the budget line (the outer boundary of the budget set) by finding two points along the line. The easiest two to find are the all-bread bundle (with $24$ to spend and bread
Figure S10.1. Indifference curves for Problem 10.6. More money left over is always good, but since cotton candy has subutility $4c - c^2$ which decreases for increasing $c$ past $c = 2$, the indifference curves bend back up: Give the consumer more than two sticks of cotton candy, and to keep her on the same indifference curve, you have to give her more money.

costing $1.20 per loaf, the consumer can buy 20 loaves of bread if she buys no cheese) and the all-cheese bundle ($24 will buy 8 kilos of cheese at $3 per kilo). These two bundles correspond to the two heavy dots, draw a straight line joining them (the budget line), fill in the triangle (the shading), and you have Figure S10.2.

Figure S10.2. Problem 10.8: A budget set.

10.9  (a) In Figure S10.3, I supply the budget set for $40 to spend and the price of wine at $10 per bottle. It is pretty clear that tangency is achieved at one bottle of wine and $30 left in the consumer’s pocket.
(b) In Figure S10.4, I supply the budget set for $40 to spend and the price of wine at $30 per bottle. We don’t achieve a tangency because we run into the constraint that the consumption of wine must be nonnegative; the best this consumer can do is to forego any wine and keep the whole $40.

10.10 See Figure S10.5 for one sort of picture. This is a topographical contour map with a peak in the middle of the consumption plane.

The slanting of the ovals means something, but it is something that is not at all implied by what was in the problem. Can you figure out what that something is, and what it would mean if the ovals were slanted in the other direction? (Hint: As we increase the amount of cotton candy, what happens to the level of consumption of fudge that is best in combination with that amount of cotton candy?)

10.11 Because the consumer would have utility \(-\infty\) if she consumed 0 of either bread or salami, we know she would consume strictly positive
amounts of both of these. Their bangs for the buck at the prices $2$ per loaf of bread and $2.50$ per kilo of salami are

\[ \frac{4}{2b} \quad \text{and} \quad \frac{0.5}{2.5s}, \]

respectively. The bang for the buck of cheese is

\[ \frac{1}{p_c(c + 1)}. \]

Hypothesize momentarily that the consumer chooses a strictly positive amount of cheese, and the equal bangs-for-the-buck rule says that these three bangs must be equal. Flipping the fractions over, this is

\[ \frac{b}{2} = 5s = p_c(c + 1). \]

We also have the budget equation $2b + p_c c + 2.5s = 10$. From the equal bangs-for-the-buck equations,

\[ 2b = 4p_c(c + 1), \quad \text{and} \quad 2.5s = 0.5p_c(c + 1), \]

so the budget equation can be rewritten

\[ 4p_c(c + 1) + p_c(c + 1) + 0.5p_c(c + 1) = 5.5p_c(c + 1) = 10, \]
or

\[ c(p_c) = \frac{10 - 5.5p_c}{5.5p_c}. \]

This is the demand curve for price \( p_c \) such that this quantity is nonnegative; when \( p_c \) exceeds 10/5.5, the demand for cheese is 0.

10.12 (a) To find the inverse demand functions, we simply take the derivatives of the “subutility” functions. So,

- The inverse demand function for bread is \( P(b) = 1/b \).
- The inverse demand function for cheese is \( P(c) = 1/(c + 3) \).
- The inverse demand function for fudge is \( P(f) = 2 - 2f \). Note that this becomes negative when \( f \) goes above 1, which is correct; to get this consumer to choose to consume more than 1 kilo of fudge, we have to pay him.

If you want demand functions, you have to invert these:

- The demand function for bread is \( D(p_b) = 1/p_b \).
- The demand function for cheese is \( D(p_c) = (1/p_c) - 3 \). Note that this becomes negative when \( p_c > 1/3 \), which means that the consumer buys no cheese at prices above $0.33 per kilo. (Either not a cheese lover or not very good cheese.)
- The demand function for fudge is \( D(p_f) = (2 - p_f)/2 \). Note that this becomes negative at \( p_f = 2 \), which means that the consumer buys no fudge at more than $2 per kilo.

(b) The expenditure on bread is a constant $1. The expenditure on cheese is \( p_c[(1/p_c) - 3] = 1 - 3p_c \), which is maximized when \( p_c = 0 \). This is a bit meaningless, as at this price, the consumer demands an infinite amount of cheese. But as the price of cheese rises, the expenditure on cheese rises toward $1. So we take $1 as the maximal level of expenditure on cheese. And expenditure on fudge is \( (2p_f - p_f^2)/2 \), which is maximized at \( p_f = 2 \), for an expenditure of $0.50. Therefore, the most this consumer spends on these three items is $2.50; as long as the consumer has more than $2.50 to start with, his expenditures on these three items leave him with some money left over, whatever their prices.

10.13 For each of these cases, the consumer’s demand function is “given” by the derivative of the subutility function. More precisely, the consumer’s inverse demand function is \( v'(x) \), where \( v' \) denotes the derivative of \( v \).
(a) So when \( v(x) = x^{1/2} \), \( v'(x) = (1/2)x^{-1/2} \). Therefore, the consumer’s inverse demand function is \( P(x) = (1/2)x^{-1/2} \), and her demand function is the inverse of this, or \( D(p) = 2/p^2 \). Note that the quantity demanded approaches infinity as price approaches 0, and it is strictly positive for all prices.

(b) When \( v(x) = 10 \ln(x + 1) \), \( v'(x) = 10/(x + 1) \). Therefore, the consumer’s inverse demand function is \( P(x) = 10/(x + 1) \). Note that, at \( x = 0 \), this is 10, which means that, at prices above \$10 \) (or whatever currency is in use for measuring money here), the consumer buys none of this good. As \( x \) goes to infinity, this stays positive. So demand is the “inverse” of this, or

\[
D(p) = \begin{cases} 
0, & \text{if } p > 10, \\
(10/p) - 1, & \text{if } 0 < p \leq 10.
\end{cases}
\]

(To get the expression \( (10/p) - 1 \), solve the equation \( p = 10/(x + 1) \) for \( x \) in terms of \( p \).)

(c) When \( v(x) = 6x - x^2 \), \( v'(x) = 6 - 2x \). Note that at \( x = 0 \), this is 6 and it hits 0 at \( x = 3 \). Hence, this consumer buys no \( x \) if its price is above \$6 \), and even if \( x \) is given away for free, she takes only 3 units of it. This is the linear demand function

\[
D(p) = \begin{cases} 
0, & \text{if } p > 6, \\
3 - (p/2), & \text{if } 0 \leq p \leq 6.
\end{cases}
\]

(d) For this subutility function \( v \), \( v'(x) = 1/x \) for \( x \leq 1 \) and \( 3 - 2x \) for \( x \geq 1 \). It is perhaps worth checking that at the critical value of \( x = 1 \), this utility function approaches 0 from both sides; it is continuous. And its derivative approaches 1 from both sides; it is smooth. The point of including this example is that no matter how high is the price, demand is positive; for \( p \geq 1 \), \( D(p) = 1/p \). But, at price \( p = 0 \), the consumer wants only \( x = 3/2 \). So we have a demand function that never hits 0, no matter how high the price but goes to a finite level as price goes to 0.

To fill in the blanks, for this sort of problem, in which utility for \( x \) and money left over has the form \( v(x) + m \):

- Demand is strictly positive no matter how high prices get if \( v'(0) = \infty \), but if \( v'(0) \) is a finite number, demand hits 0 when price rises to \( v'(0) \).
- Demand approaches \( \infty \) as the price declines to 0 if \( v' > 0 \) no matter how large \( x \) becomes. But if \( v' \) hits 0 at some finite level of \( x \), then demand stops at that level when the price goes to 0.
10.14 (As you read this solution, keep thinking about analogies to the franchise-fee discussion from Chapter 9.) Suppose the offer from the firm is: Pay fee $F$ and you can buy as much as you want at a per-unit price of $p$ each. If the consumer pays the up-front fee, it becomes a sunk cost, and her level of demand is that value $x_1$ that solves $v'_1(x_1) = p$. (This assumes that $v'_1(0) > p$; if not, the consumer would demand 0 and, so, would never pay an up-front fee.) The net impact of this in terms of the consumer’s overall utility is

$$v_1(x_1) - v_1(0) - px_1 - F;$$

that is, she gains utility $v_1(x_1) - v_1(0)$, but her money-left-over stock is decreased by $px_1 + F$. So she will pay the fee and buy this amount as long as this net impact is positive, or $v_1(x_1) - v_1(0) - px_1 - F \geq 0$. This means that, for a given $p$ and corresponding $x_1$, the biggest up-front fee the firm can charge (and get acceptance of their offer) is $F = v_1(x_1) - v_1(0) - px_1$, and since their overall take is $F + px_1$, this means that the firm’s net take in revenue (if the $p$ they pick induces the consumer to choose $x_1$) is

$$v_1(x_1) - v_1(0).$$

(The units of $v_i$ are “utils,” so it may seem strange that the dollar take of the firm is being measured in utility units. But remember that, with a money-left-over utility function, utility units are denominated in dollar amounts.)

Of course, the firm has to pay to produce the $x_i$ goods, and as their marginal cost is (per the problem) a fixed $c$ per unit, their overall profit-contribution from this consumer is

$$v_i(x_i) - v_i(0) - cx_i.$$  

This is maximized where $v'_i(x_i) = c$. (Does this begin to sound familiar with the discussion in Chapter 9? If not, re-review Chapter 9.) So, in the form of a fixed up-front entry or access fee $F$ and a per-unit charge $p$, the profit-maximizing deal for the firm to offer this consumer is

$$F = v_1(x_1) - v_1(0) - cx_1 \quad \text{and} \quad p = c, \quad \text{where } x_i \text{ solves } v'_i(x_i) = c.$$  

Note that, if this firm is dealing with multiple consumers in this fashion, the per-unit price it charges is the same for all, namely its marginal cost, while the entry fee is customized for different consumers: Consumers pay more
up front if their total utility gain (measured in dollar terms) from consuming the good is higher. (The parallel with last chapter should now be obvious.)

And, in terms of take-it-or-leave-it offers: If the firm somehow “knows” an individual consumer’s sub-utility function $v_1$, it should (1) find the value $x_1$ that satisfies $v_1'(x_1) = c$ and tell the consumer: “You can have $x_1$ units of the good if you make a one-time payment of $v_1(x_1) - v_1(0)$ [or a little less, to induce acceptance], take it or leave it.”